Philip Lee

Former IB Student at Exeter College, Devon

The Problem (unofficial):

Calculating the difference between equally powered numbers

My solution:

$$\mathrm{x}^{\mathrm{n}} - \mathrm{y}^{\mathrm{n}} = (\mathrm{x} - \mathrm{y})\left(\sum_{\theta=0}^{\mathrm{c}-1} \mathrm{x}^{\mathrm{c}-1-\theta}\mathrm{y}^{\theta}\right)\prod_{u=1}^{\lambda}\left(x^{2^{\mathrm{u}-1}c} + y^{2^{u-1}c}\right)$$

where $n = 2^{\lambda}c$ and $n, \lambda \in \mathbb{N}$.

In this equation x and y are the rational numbers being powered, and n is the power which is written as the product of a natural number, c, and 2 to the natural power, λ.

For example:

If I want x and y to be raised to the power of 8, 8 must first be plugged into the expression for n.

So: $8 = 2^{\lambda}c$. Now I can choose what c and λ are, as long as they satisfy this expression. So I could use $c = 2, \lambda = 2$ or $c = 4, \lambda = 1$ etc. I'll take $c = 2, \lambda = 2$.

So then I plug these values into my equation:

$$x^8 - y^8 = (x - y)\left(\sum_{\theta=0}^{2-1} x^{2-1-\theta}y^{\theta}\right)\prod_{u=1}^{2}\left(x^{2^{\mathrm{u}-1}\times 2} + y^{2^{u-1}\times 2}\right)$$

Then by expansion:

$$x^8 - y^8 = (x - y)\ (x + y)(x^2 + y^2)(x^4 + y^4)$$

So

$$\begin{aligned} x^8 - y^8 &= (x^2 - y^2)(x^6 + x^2y^4 + x^4y^2 + y^6) \\ &= x^8 + x^4y^4 + x^6y^2 + x^2y^6 - x^6y^2 - x^2y^6 - x^4y^4 - y^8 \\ &= x^8 - y^8 \end{aligned}$$

You'll find this works for any whole number value of n and any positive real number of x and y.

Some properties of the equation:

Each power of x and y is part of a series of doubling powers. Notice:

$$x^2 - y^2 = (x - y)(x + y)$$

and

$$x^4 - y^4 = (x - y)(x + y)(x^2 + y^2)$$
$$= (x^2 - y^2)(x^2 + y^2)$$

and

$$x^8 - y^8 = (x - y)(x + y)(x^2 + y^2)(x^4 + y^4)$$
$$= (x^4 - y^4)(x^4 + y^4)$$

And so on, such that

$$x^n - y^n = (x^{n-1} - y^{n-1})(x^{n-1} + y^{n-1})$$

Hence how each powered difference is the previous powered difference multiplied by the previous power as a sum. In this way each power is linked to its series in factors of 2. So if one difference in the series is known, the others may be calculated by this method.

This also links to the fact that if $n = c$ (so $\lambda = 0$) then

$$x^n - y^n = (x - y)\left(\sum_{\theta=0}^{n-1} x^{(n-1-\theta)} y^{\theta}\right)$$

which was the key to this particular solution to the difference between powered numbers.

ADDITIONAL: these equations were originally sought after in order to improve ease of mental calculations and this has proved to have been achieved, especially when considering the value of a squared number when a close square is known.

For example: 20^2 is known to be 400. I want to calculate 22^2.

(Here, $x = 22, y = 20, n = 2$)

So rearrange the equation for $n = 2$ (previously stated) to give:

$$x^2 = y^2 + x(x - y) + y(x - y)$$

(This was actually the original equation found which led to the general statement)

Therefore inputting the values gives:

$22^2 = 400 + 22(2) + 20(2) = 484$